Quantum Mechanics II: Examples

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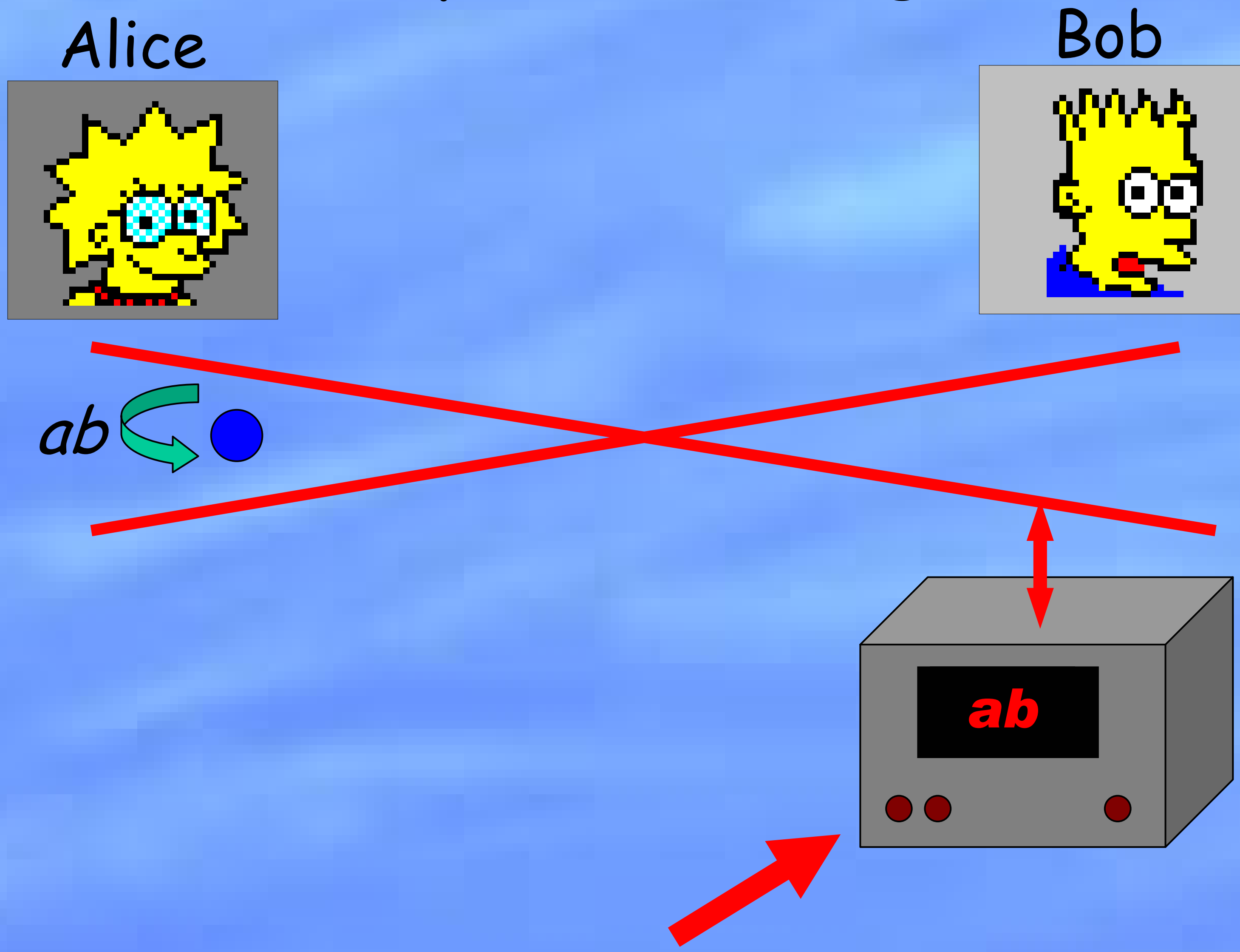
Goals:

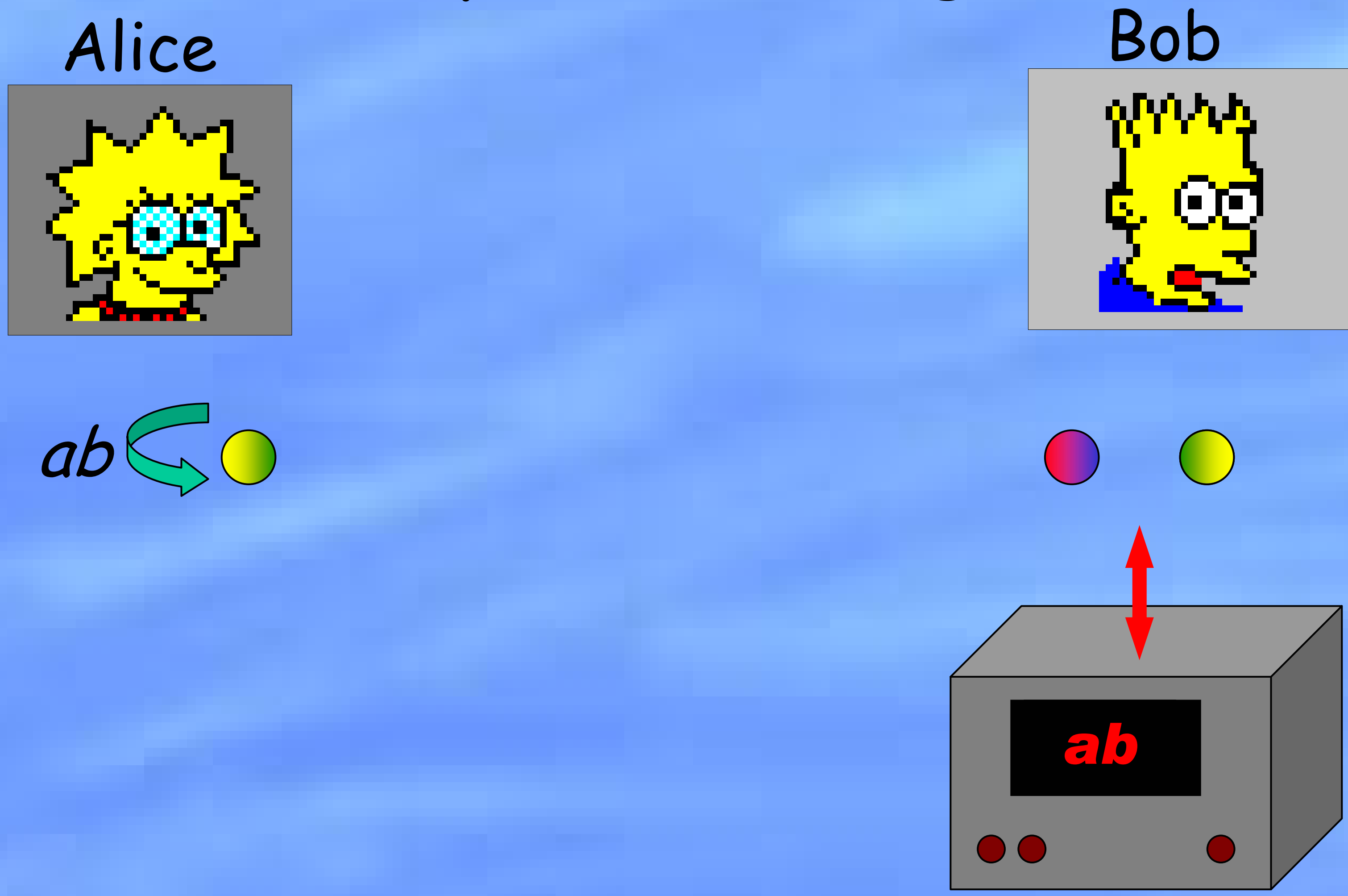
1. To apply the principles introduced in the last lecture to some illustrative examples: superdense coding, and quantum teleportation.
2. Revised form of postulates 2 (dynamics) and 3 (measurement).
3. Introduce more elements of the Dirac notation.
4. Discuss the philosophy underlying quantum information science.

Superdense coding



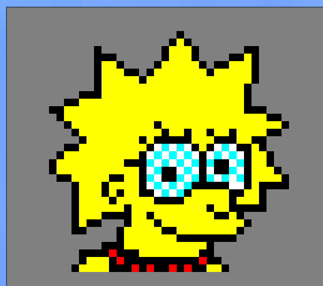
Theorist's impression of a measuring device

Superdense coding

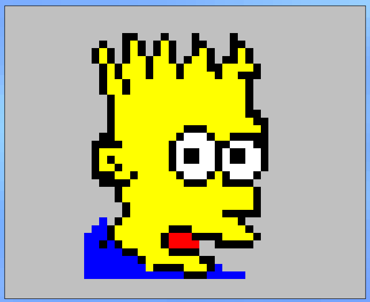


Superdense coding

Alice

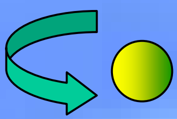


Bob

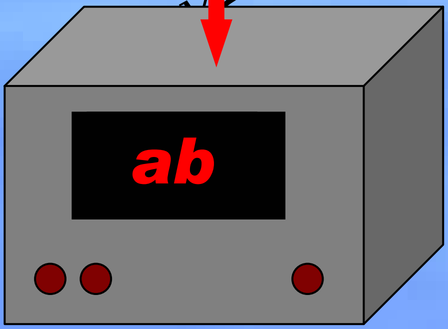


$$X|0\rangle = |1\rangle; \quad X|1\rangle = |0\rangle$$

$$Z|0\rangle = |0\rangle; \quad Z|1\rangle = -|1\rangle$$

ab 

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



00 : Apply I $\quad \frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

01 : Apply Z $\quad \frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \frac{|00\rangle - |11\rangle}{\sqrt{2}}$

10 : Apply X $\quad \frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \frac{|10\rangle + |01\rangle}{\sqrt{2}}$

11 : Apply XZ $\quad \frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \frac{|10\rangle - |01\rangle}{\sqrt{2}}$

Superdense coding can be viewed as a statement about the interchangeability of physical resources.

1 ebit + 1 qubit of communication $\geq$ 2 bits of classical communication

Worked exercise: Could Alice and Bob still communicate two bits using the superdense coding protocol if the initial state shared by Alice and Bob was $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$?

Revised measurement postulate

Recall postulate 3: If we measure $|\psi\rangle$ in an orthonormal basis $|e_1\rangle, \ldots, |e_d\rangle$, then we obtain the result j with probability

$$P(j) = \left|\langle e_j|\psi\rangle\right|^2 .$$

The measurement disturbs the system, leaving it in a state $|e_j\rangle$ determined by the outcome.

Problem: Imagine we measure a quantum system, A, in the orthonormal basis $|e_1\rangle, \ldots, |e_{d_A}\rangle$.

Suppose system A is part of a larger system, consisting of two components, A and B.

How should we describe the effect of the measurement on the larger system?

Revised measurement postulate

Recall postulate 3: If we measure $|\psi\rangle$ in an orthonormal basis $|e_1\rangle,\ldots,|e_d\rangle$, then we obtain the result j with probability

$$P(j) = \left|\langle e_j|\psi\rangle\right|^2 .$$

The measurement disturbs the system, leaving it in a state $|e_j\rangle$ determined by the outcome.

The revised postulate replaces the orthogonal states $|e_1\rangle,\ldots,|e_d\rangle$ with a complete set of orthogonal subspaces $V_1,\ldots,V_m$.

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_m .$$

Example: $|\psi\rangle = \left(\alpha|e_1\rangle + \beta|e_2\rangle\right) + \gamma|e_3\rangle$

$$\mathrm{sp}\left(|e_1\rangle,|e_2\rangle,|e_3\rangle\right) = \mathrm{sp}\left(|e_1\rangle,|e_2\rangle\right) \oplus \mathrm{sp}\left(|e_3\rangle\right)$$

A general measurement can be thought of as asking the question "which of the subspaces $V_1,\ldots,V_m$ are we in?"

Revised measurement postulate

Recall postulate 3: If we measure $|\psi\rangle$ in an orthonormal basis $|e_1\rangle, \ldots, |e_d\rangle$, then we obtain the result j with probability

$$P(j) = \left|\langle e_j|\psi\rangle\right|^2.$$

The measurement disturbs the system, leaving it in a state $|e_j\rangle$ determined by the outcome.

Mathematically, it is convenient to describe the subspaces $V_1, \ldots, V_m$ in terms of their corresponding projectors, $P_1, \ldots, P_m$.

Example: The projector P onto $\mathrm{sp}(|e_1\rangle, |e_2\rangle)$ acts as

$$P\left(\alpha|e_1\rangle + \beta|e_2\rangle + \gamma|e_3\rangle\right) = \alpha|e_1\rangle + \beta|e_2\rangle$$

In general, the projector P onto a subspace V acts as the identity on that subspace, and annihilates everything orthogonal to V.

Revised measurement postulate

Let $P_1,\ldots,P_m$ be a set of projectors onto a complete set of orthogonal subspaces of state space.

$$\sum_j P_j = I; \qquad P_j P_k = \delta_{jk} P_j$$

This set of projectors defines a measurement.

If we measure $|\psi\rangle$ then we get outcome j with probability $\Pr(j) = \langle\psi|P_j|\psi\rangle$.

The measurement unavoidably disturbs the system, leaving it in the post-measurement state $\dfrac{P_j|\psi\rangle}{\sqrt{\langle\psi|P_j|\psi\rangle}}$

Example: A two-outcome measurement on a qutrit

A general state of a qutrit may be written $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$.

P_1 projects onto $\mathrm{sp}(|0\rangle, |1\rangle)$; and

P_2 projects onto $\mathrm{sp}(|2\rangle)$.

$$\Pr(1) = \langle\psi|P_1|\psi\rangle = \begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} \bullet \begin{bmatrix}\alpha\\ \beta\\ 0\end{bmatrix} = |\alpha|^2 + |\beta|^2$$

$$\Pr(2) = \langle\psi|P_2|\psi\rangle = |\gamma|^2$$

$$|\psi_1'\rangle = \frac{P_1|\psi\rangle}{\sqrt{\langle\psi|P_1|\psi\rangle}} = \frac{\alpha|0\rangle + \beta|1\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}} \qquad |\psi_2'\rangle = \frac{P_2|\psi\rangle}{\sqrt{\langle\psi|P_2|\psi\rangle}} = \frac{\gamma|2\rangle}{|\gamma|} \sim |2\rangle$$

Example: Measuring the first of two qubits

Suppose we want to perform a measurement in the basis $|e_1\rangle, |e_2\rangle$ for the first of two qubits.

The rule is to first form the corresponding projectors P_1, P_2 onto the state space of that qubit, and then to tensor them with the identity on the second qubit, obtaining $P_1 \otimes I$ and $P_2 \otimes I$.

Example: If the state of two qubits is

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

then measuring the first qubit in the computational basis gives the result 0 with probability

$$\begin{aligned}
\Pr(0) &= \langle\psi|(P_0 \otimes I)|\psi\rangle \\
&= (\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle) \bullet (\alpha_{00}|00\rangle + \alpha_{01}|01\rangle) \\
&= |\alpha_{00}|^2 + |\alpha_{01}|^2
\end{aligned}$$

Example: Measuring the first two of three qubits

Suppose we have three qubits in the state

$$\alpha|e_1\rangle|a\rangle+\beta|e_2\rangle|b\rangle+\gamma|e_3\rangle|c\rangle+\delta|e_4\rangle|d\rangle.$$

$|e_1\rangle,|e_2\rangle,|e_3\rangle,|e_4\rangle$ is an orthonormal basis for the state space of the first two qubits.

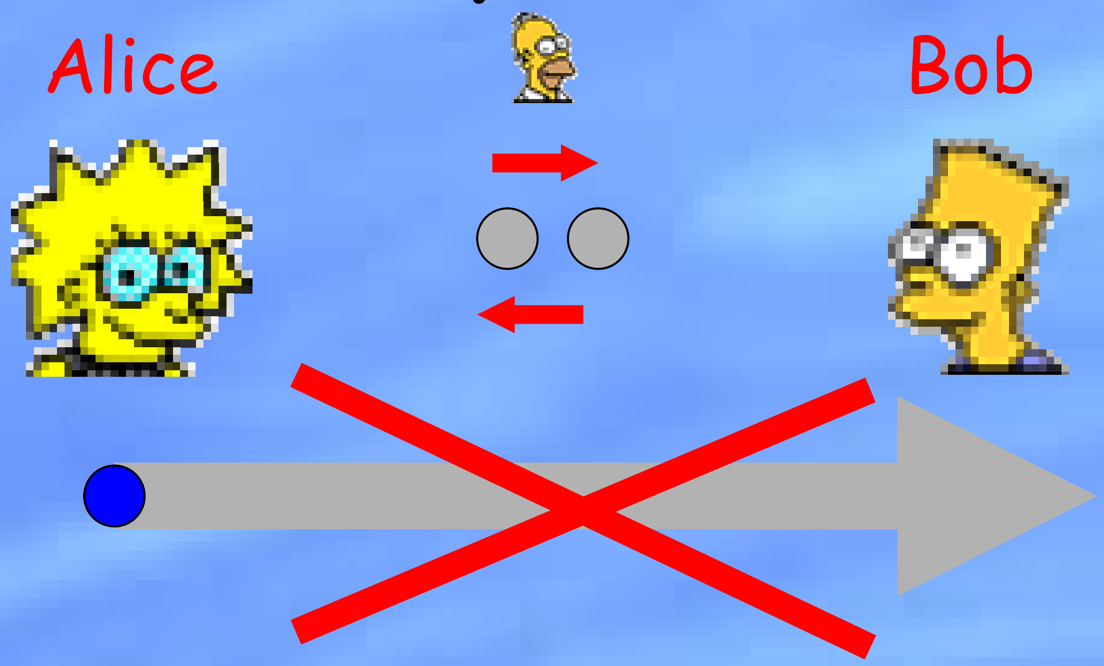
$|a\rangle,|b\rangle,|c\rangle,|d\rangle$ are normalized states of the third qubit.

Measuring the first two qubits in the basis $|e_1\rangle,|e_2\rangle,|e_3\rangle,|e_4\rangle$ gives the result 1 with probability

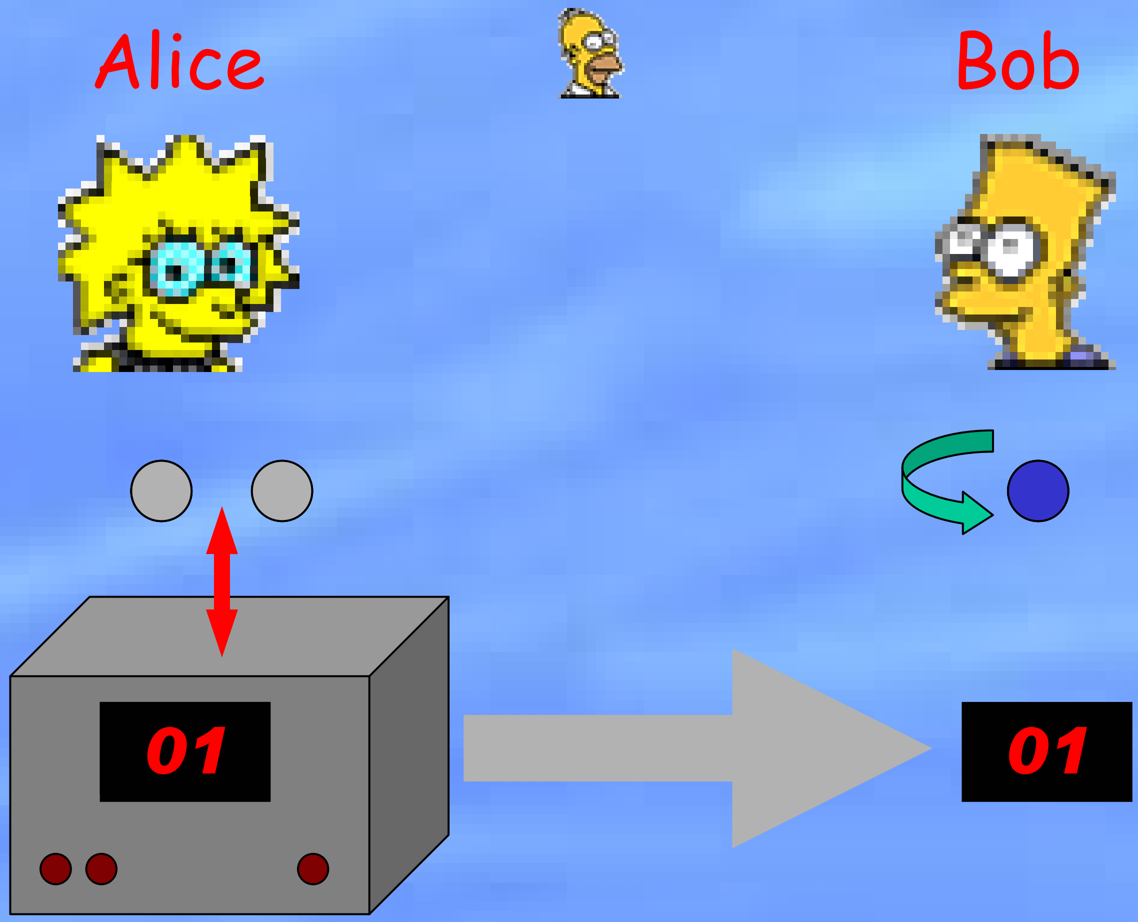
$$\begin{aligned}\Pr(1) &= \langle\psi|\left(P_1\otimes I\right)|\psi\rangle\\ &=\langle\psi|\left(\alpha|e_1\rangle|a\rangle\right)\\ &=|\alpha|^2\end{aligned}$$

Post-measurement state is $|e_1\rangle|a\rangle$.

Teleportation



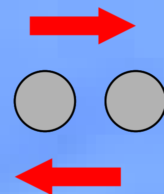
Teleportation



Teleportation

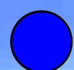
Alice

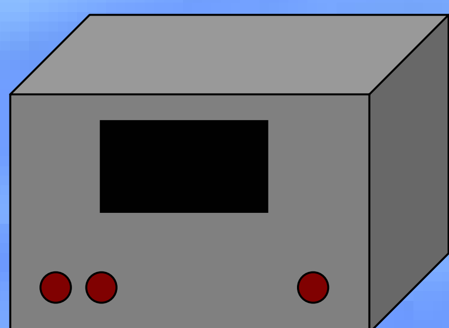
Bob



$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$\alpha|0\rangle + \beta|1\rangle$$





$$\left(\alpha|0\rangle + \beta|1\rangle\right)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

$$= \frac{\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle}{\sqrt{2}}$$

$$|00\rangle = \frac{1}{\sqrt{2}}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right)$$

$$|10\rangle = \frac{1}{\sqrt{2}}\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)$$

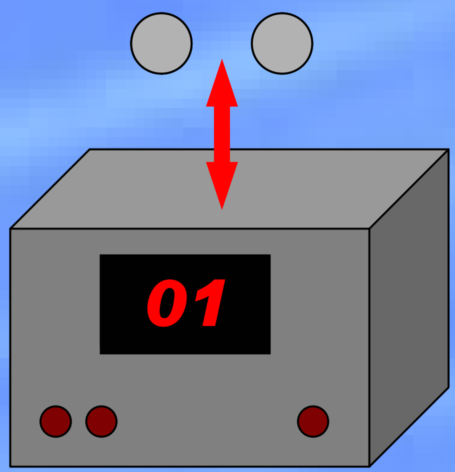
$$|01\rangle = \frac{1}{\sqrt{2}}\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)$$

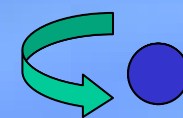
$$|11\rangle = \frac{1}{\sqrt{2}}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right)$$

Teleportation

Alice

Bob





$$= \frac{1}{2}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\alpha|0\rangle + \beta|1\rangle\right) \xrightarrow{I} \left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$+ \frac{1}{2}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right)\left(\alpha|0\rangle - \beta|1\rangle\right) \xrightarrow{Z} \left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$+ \frac{1}{2}\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)\left(\alpha|1\rangle + \beta|0\rangle\right) \xrightarrow{X} \left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$+ \frac{1}{2}\left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)\left(\alpha|1\rangle - \beta|0\rangle\right) \xrightarrow{ZX} \left(\alpha|0\rangle + \beta|1\rangle\right)$$

Teleportation can be viewed as a statement about the interchangeability of physical resources.

1 ebit + 2 classical bits of communication $\geq$ 1 qubit of communication

Compare with superdense coding:

1 ebit + 1 qubit of communication $\geq$ 2 bits of classical communication

1 qubit of communication = 2 bits of communication (Mod 1 ebit)

The fundamental question of information science

1. Given a physical resource – energy, time, bits, space, entanglement; and
2. Given an information processing task – data compression, information transmission, teleportation; and
3. Given a criterion for success;

We ask the question:

How much of 1 do I need to achieve 2, while satisfying 3?

Pursuing this question in the quantum case has led to, and presumably will continue to lead to, interesting new information processing capabilities.

"How to write a quant-ph"

Are there any fundamental scientific questions that can be addressed by this program?

What fundamental problems are addressed by quantum information science?

You

Your challenger

Knowing the rules $\neq$ Understanding the game

Knowing the rules of quantum mechanics

$\neq$

Understanding quantum mechanics

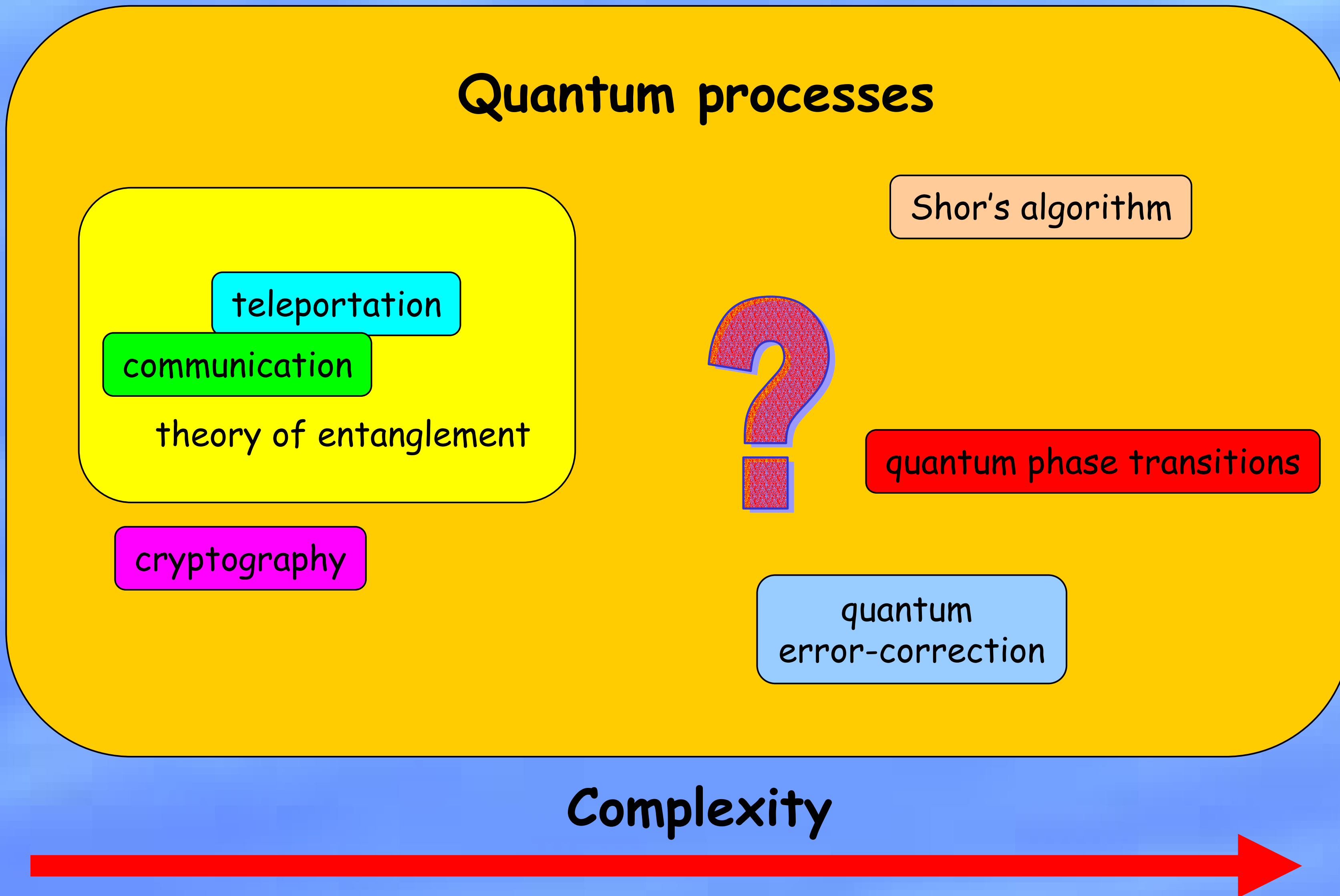
What high-level principles are implied by quantum mechanics?

Robert B. Laughlin, 1998 Nobel Lecture

“I give my class of extremely bright graduate students, who have mastered quantum mechanics but are otherwise unsuspecting and innocent, a take-home exam in which they are asked to deduce superfluidity from first principles.
There is no doubt a special place in hell being reserved for me at this very moment for this mean trick, for the task is impossible.

Superfluidity, like the fractional quantum Hall effect, is an emergent phenomenon – a low-energy collective effect of huge numbers of particles that cannot be deduced from the microscopic equations of motion in a rigorous way and that disappears completely when the system is taken apart (Anderson, 1972)”

Quantum information science as an approach to the study of complex quantum systems



A few quanta of miscellanea

The "outer product" notation

The spectral theorem - diagonalizing Hermitian matrices

Historical digression on measurement

The trace operation

Quantum dynamics in continuous time: an alternative form of the second postulate

Outer product notation

Let $|\psi\rangle$ and $|\phi\rangle$ be vectors.

Define a linear operation (matrix) $|\psi\rangle\langle\phi|$ by

$$|\psi\rangle\langle\phi|\big(|\gamma\rangle\big) \equiv |\psi\rangle\langle\phi|\gamma\rangle$$

Example: $|1\rangle\langle 0|\big(\alpha|0\rangle + \beta|1\rangle\big) \equiv |1\rangle\alpha = \alpha|1\rangle$

Connection to matrices:

If $|a\rangle = \sum_j a_j|j\rangle$, and $|b\rangle = \sum_j b_j|j\rangle$ then $|a\rangle\langle b|k\rangle = b_k^*|a\rangle$.

But $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}\begin{bmatrix} b_1^* & b_2^* & \cdots \end{bmatrix}\begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix} = b_k^*|a\rangle$.

Thus $|a\rangle\langle b| = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}\begin{bmatrix} b_1^* & b_2^* & b_3^* \end{bmatrix}$.

Outer product notation

Example: $|0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Example: $|1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Example: $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$

Example: $|0\rangle\langle 1| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Example: $|1\rangle\langle 0| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Example: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$

Exercise: Find an outer product representation for Y.

Outer product notation

One of the advantages of the outer product notation is that it provides a convenient tool with which to describe projectors, and thus quantum measurements.

Recall: The projector P onto $\mathrm{sp}(|e_1\rangle, |e_2\rangle)$ acts as

$$P(\alpha|e_1\rangle + \beta|e_2\rangle + \gamma|e_3\rangle) = \alpha|e_1\rangle + \beta|e_2\rangle$$

This gives us a simple explicit formula for P, since

$$(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|)(\alpha|e_1\rangle + \beta|e_2\rangle + \gamma|e_3\rangle) = \alpha|e_1\rangle + \beta|e_2\rangle$$

More generally, the projector onto a subspace spanned by orthonormal vectors $|e_1\rangle, \ldots, |e_m\rangle$ is given by

$$P = \sum_j |e_j\rangle\langle e_j|.$$

Exercise: Suppose $|e_1\rangle, \ldots, |e_d\rangle$ is an orthonormal basis for state space. Prove that $I = \sum_j |e_j\rangle\langle e_j|$.

Exercise: Prove that $|a\rangle\langle b|^\dagger = |b\rangle\langle a|$.

The spectral theorem

Theorem: Suppose A is a Hermitian matrix, $A^\dagger = A$. Then A is **diagonalizable**,

$$A = U\text{diag}(\lambda_1,\ldots,\lambda_d)U^\dagger,$$

where U is unitary, and $\lambda_1,\ldots,\lambda_d$ are the eigenvalues of A.

But $\text{diag}(\lambda_1,\ldots,\lambda_d) = \sum_j \lambda_j |j\rangle\langle j|$.

Thus $A = \sum_j \lambda_j |e_j\rangle\langle e_j|$, where $|e_j\rangle \equiv U|j\rangle$ is the λ_j eigenvector of A, $A|e_j\rangle = \lambda_j |e_j\rangle$.

$A = \sum_k \lambda_k P_k$, where P_k is the projector onto the λ_k eigenspace of A.

Examples of the spectral theorem

Example: $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$

Example: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvectors $|\pm\rangle \equiv \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$, with corresponding eigenvlaues ± 1.

$$|+\rangle\langle +| - |-\rangle\langle -| = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Historical digression: the measurement postulate formulated in terms of "observables"

Our form: A complete set of projectors P_j onto orthogonal subspaces. Outcome j occurs with probability

$$\Pr(j) = \langle\psi|P_j|\psi\rangle.$$

The corresponding post-measurement state is

$$\frac{P_j|\psi\rangle}{\sqrt{\langle\psi|P_j|\psi\rangle}}.$$

Old form: A measurement is described by an observable, a Hermitian operator M, with spectral decomposition

$$M = \sum_j \lambda_j P_j.$$

The possible measurement outcomes correspond to the eigenvalues λ_j, and the outcome λ_j occurs with probability

$$\Pr(j) = \langle\psi|P_j|\psi\rangle.$$

The corresponding post-measurement state is

$$\frac{P_j|\psi\rangle}{\sqrt{\langle\psi|P_j|\psi\rangle}}.$$

An example of observables in action

Example: Suppose we "measure Z".

Z has spectral decomposition $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$, so this is just like measuring in the computational basis, and calling the outcomes "1" and "-1", respectively, for 0 and 1.

Exercise: Find the spectral decomposition of $Z \otimes Z$. Show that measuring $Z \otimes Z$ corresponds to measuring the parity of two qubits, with the result +1 corresponding to even parity, and the result -1 corresponding to odd parity.

Exercise: Suppose we measure the observable M for a state $|\psi\rangle$ which is an eigenstate of that observable. Show that, with certainty, the outcome of the measurement is the corresponding eigenvalue of the observable.

The trace operation

$$\mathrm{tr}(A) \equiv \sum_j A_{jj}$$

Examples: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\mathrm{tr}(X) = 0;$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathrm{tr}(I) = 2.$

Cyclicity property: $\mathrm{tr}(AB)=\mathrm{tr}(BA)$.

$$\mathrm{tr}(AB) = \sum_j (AB)_{jj} = \sum_{jk} A_{jk} B_{kj} = \sum_{jk} B_{kj} A_{jk} = \sum_k (BA)_{kk} = \mathrm{tr}(BA)$$

Exercise: Prove that $\mathrm{tr}(|a\rangle\langle b|)=\langle a|b\rangle$.

An alternative form of postulate 2

Postulate 2: The evolution of a closed quantum system is described by a unitary transformation.

$$|\psi'\rangle = U|\psi\rangle$$

But quantum dynamics occurs in continuous time!

An alternate form of postulate 2

The evolution of a closed quantum system is described by Schroedinger's equation:

$$i\frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

where H is a constant Hermitian matrix known as the Hamiltonian of the system.

The eigenvectors of H are known as the energy eigenstates of the system, and the corresponding eigenvalues are known as the energies.

Example: $H = \omega X$ has energy eigenstates $(|0\rangle + |1\rangle)/\sqrt{2}$ and $(|0\rangle - |1\rangle)/\sqrt{2}$, with corresponding energies $\pm\omega$

Connection to old form of postulate 2

The solution of Schroedinger's equation is $|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle$

$$U \equiv \exp(-iHt) \qquad |\psi'\rangle = U|\psi\rangle$$